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Heat transfer to a quadratic shear profile

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An operational method is used to obtain an exact solution for the heat transfer from a surface on which the temperature is prescribed to a stream in which the velocity profile is given by

$$u = (\mu_w \rho_w)^{-1}\left[\tau_w \eta + \left(\frac{1}{2\rho_w}\frac{dp}{dx}\right)\eta^2\right],$$

where

$$\eta = \int_0^y \rho\, dy$$

and other symbols have their usual meanings. The solution is expanded for small and large values of a dimensionless parameter proportional to $\sigma^{\frac{1}{4}}\tau_w/(dp/dx)^{\frac{2}{3}}$: the leading term when this is large is precisely Lighthill's (1950) expression for heat transfer at high Prandtl number, and that when it is small corresponds to Liepmann's (1958) expression for heat transfer at a separation point.

Particular attention is given to the case of heat transfer from a small element of length l maintained at a constant temperature ΔT above that of the surrounding adiabatic wall: this represents the gauge for skin friction measurement described by Bellhouse & Schultz (1966) and Brown (1967*a*). It is shown that the Nusselt number for heat transfer from such an element is of the form $\alpha^{\frac{1}{3}}f(\beta/\alpha^{\frac{4}{3}})$, where

$$\alpha = \sigma\rho_w\tau_w\frac{l^2}{\mu_w^2}, \quad \beta = \left(\frac{\sigma\rho_w}{\mu_w^2}\right)\frac{dp}{dx}l^3,$$

and expansions for small and large values of $\beta/\alpha^{\frac{4}{3}}$ are given. Over the whole range both are adequately represented for experimental purposes by the equation

$$c_1\alpha + \frac{c_2\beta}{Nu} = Nu^3,$$

which has a form suggested by consideration of the integral approximation of Curle (1962). The experimental application of the results to both laminar and turbulent flows is discussed.

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1. Introduction

Lighthill's (1950) investigation of the heat transfer to a laminar stream from a heated wall has been the starting-point for much further work. Lighthill solved the boundary-layer form of the diffusion equation, assuming a linear velocity profile

$$u = y\tau_w(x)/\mu, \tag{1.1}$$

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τ_w being the local skin friction and μ the viscosity. This led to a convolution integral relating the heat transfer at the wall to the distributions of shear stress and wall temperature. The velocity profile is certainly of this form close enough to the wall, and if the thermal conductivity λ of the stream is sufficiently small, i.e. the Prandtl number $\sigma = \mu C_p/\lambda$ sufficiently large, heat from the wall will penetrate only to a region where the velocity profile is well represented by (1.1). Lighthill's result is therefore asymptotically correct for large σ whatever the pressure gradient. By means of an elegant dimensional argument, Liepmann (1958) subsequently derived an expression identical in structure to Lighthill's from an integral form of the energy equation, using an empirical assumption of similarity between the enthalpy and velocity profiles to evaluate certain parameters, and showed that reasonable assumptions in this regard led to close agreement with Lighthill's exact analysis. Liepmann went on to consider the effect of a pressure gradient, which would make itself felt at finite Prandtl number by means of the second term

$$\frac{1}{2\mu}\left(\frac{dp}{dx}\right)y^2 \tag{1.2}$$

in the velocity profile. For flow near a separation point where (1.2) dominates over (1.1), he obtained an expression for the heat transfer in which $[\sigma(dp/dx)]^{\frac{1}{4}}$ occupies the place taken by $(\sigma\tau_w)^{\frac{1}{3}}$ in the Lighthill analysis. A somewhat similar analysis was also given by Spalding (1958). Subsequently Curle (1961), using an approach modelled on that of Liepmann, has shown how Lighthill's method may be considerably improved in accuracy over the whole range of τ_w and dp/dx, the criterion being agreement with exact solutions of the *full* boundary-layer equations for flows of the Falkner–Skan type. This is secured by giving optimum values to two constants a and b occurring in his analysis. The details of all three methods, and other related work, are contained in Curle (1962, chapter 6).

Recently Bellhouse & Schultz (1966) have described a technique based on the earlier work of Ludwieg (1950) for measuring skin friction by means of a surface element maintained at a temperature above that of the stream. The heat transfer from the element should be proportional to $\tau_w^{\frac{1}{3}}$ according to Lighthill's theory, and this was well borne out experimentally for work in a zero pressure gradient. One of us (Brown 1967 *a*, *b*) has gone on to apply the same technique in a pressure gradient, and the need to relate the Nusselt number of the element as accurately as possible to the local values of τ_w and dp/dx has provided the stimulus for the work reported in the present paper.

Instead of approximating in the integrated form of the energy equation, we go back to the exact equation, and apply analysis similar to that used by Lighthill, but retaining both linear and quadratic terms in the velocity profile. The resulting equation can be solved exactly if the shear stress and pressure gradient vary in such a way that the ratio

$$\tau_w^3/(dp/dx)^2 \tag{1.3}$$

is constant. This requirement puts a limit on the applicability of the results: they can be used only when the extent of the heated region of wall is small compared with the length scale for changes in this ratio. For the small heated elements in question, whose lengths are in the order of 1 mm., this is an unimportant restriction. The results are of course exact for a Poiseuille–Couette flow with constant τ_w and dp/dx.

In place of the Airy equation to which Lighthill reduced the energy equation, we obtain a hypergeometric equation, and Lighthill's result is obtained by an asymptotic procedure based on the method of steepest descents for the limit when
$$\tau_w^3/(dp/dx)^2 \to \infty;$$
the role played by Prandtl number is then clearly seen, for the heat transfer is obtained as an expansion in powers of $(dp/dx)/\tau_w^{\frac{3}{2}}\sigma^{\frac{1}{3}}$. A result similar in form to Liepmann's is found in the opposite limit
$$\tau_w^3/(dp/dx)^2 \to 0.$$
These results are found in §§2 and 3, for a general distribution of wall temperature $T_w(x)$. The analysis is carried out for a general compressible fluid with constant Prandtl number. We assume (as do all the authors cited) that heat from the wall is diffused as a passive scalar, without affecting the velocity and density distributions in the boundary layer. This implies that $\Delta T/T_{aw} \ll 1$, where ΔT is the temperature of the heated part of the wall above the adiabatic wall temperature T_{aw}. For a typical low-speed experiment in air, ΔT would be 80 °K, giving a ratio of the order of $\frac{1}{4}$, but in fact the situation is much better than this because the excess temperature falls off very rapidly with distance from the wall. For measurements in an arterial blood flow, $\Delta T = 5$ °K would be a more typical value.

We then examine the particular case in which heating takes place only over a small length l of the surface, representing the skin friction gauge, and obtain in §4 expressions for the Nusselt number for heat transfer from the gauge in terms of dimensionless parameters
$$\alpha = \frac{\sigma\rho_w\tau_w l^2}{\mu_w^2}, \quad \beta = \frac{\sigma\rho_w}{\mu_w^2}\left(\frac{dp}{dx}\right)l^3,$$
for small and large values of the ratio $\beta/\alpha^{\frac{3}{2}}$. The results in the two limits are of the same form as would have been found from an integral solution based on Curle's work, but the constants are slightly different. In §5 we calculate the fraction of the heat from the gauge which crosses planes at various heights from the wall, in order to judge the usefulness of such a gauge for measuring skin friction in a turbulent boundary layer, to which equation (1.1) would apply in the sublayer. The paper concludes with a discussion in §6 of the experimental application of these results.

2. The energy equation

The energy equation for a viscous compressible gas is
$$\rho C_p \frac{dT}{dt} = \frac{dp}{dt} + \operatorname{div}(\lambda \operatorname{grad} T) + \Phi, \tag{2.1}$$
where Φ is the dissipation function, λ the thermal conductivity, and
$$\frac{d}{dt} \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

As stated in the introduction, we wish to discuss the passive diffusion of heat from a small element in a wall that is otherwise adiabatic, and will suppose that the quantity of heat, or the proportional rise in temperature produced by it, is insufficient to change the distribution of density and therefore of pressure and velocity in the flow field. We can then regard the full temperature field as being made up by the linear superposition of the distribution appropriate to the adiabatic wall, which is coupled to the velocity field, and satisfies (2.1), and the additional temperature generated by the element alone, which satisfies

$$\rho C_p \frac{dT}{dt} = \operatorname{div}(\lambda \operatorname{grad} T). \tag{2.2}$$

In the boundary-layer approximation, the right-hand side is replaced by

$$\frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right). \tag{2.3}$$

We shall make this approximation for the present, but it will require re-examination in §4.4 when we discuss heat transfer from a gauge of very small streamwise extent, for which case the x-derivative on the right of (2.2) may not be negligible.

Lighthill (1950) then makes the von Mises transformation in which the independent variables are x and the stream function ψ. The transformation, stated by Lighthill for incompressible flow, applies equally well in a compressible flow, for which ψ is defined by

$$\rho u = \partial\psi/\partial y, \quad \rho v = -\partial\psi/\partial x, \tag{2.4}$$

if the standard assumptions

$$\left.\begin{aligned} C_p = \text{constant}, \quad \mu C_p/\lambda \equiv \sigma = \text{constant},\\ \mu\rho \text{ independent of } y, = \mu_w(x)\rho_w(x) \end{aligned}\right\} \tag{2.5}$$

are made. Equation (2.2) becomes

$$\frac{\partial T}{\partial x} = \frac{\rho_w \mu_w}{\sigma} \frac{\partial}{\partial \psi}\left(u \frac{\partial T}{\partial \psi}\right). \tag{2.6}$$

By differentiating the boundary-layer momentum equation, the velocity profile can be expanded as a Taylor series in the Howarth variable

$$\eta = \int_0^y \rho dy \tag{2.7}$$

in the form

$$u = (\rho_w \mu_w)^{-1}[\tau_w \eta + (1/2\rho_w)(dp/dx)\eta^2 + \ldots]. \tag{2.8}$$

We now insert the first two terms of this expression for u into the energy equation (2.6). For this purpose it is convenient to introduce a further independent variable Y in place of ψ by writing

$$\eta = (\rho_w \tau_w/(dp/dx))(Y-1). \tag{2.9}$$

In terms of Y,

$$u = \frac{\tau_w^2}{2\mu_w(dp/dx)}(Y^2-1), \quad \psi = \frac{\rho_w \tau_w^3}{2\mu_w(dp/dx)^2}(\tfrac{1}{3}Y^3 - Y). \tag{2.10}$$

We may also replace x by a dimensionless variable X by means of

$$dX = (4/\sigma)(\rho_w \tau_w/\mu_w^2)^{\frac{1}{2}}\,dx. \tag{2.11}$$

Substitution of these expressions transforms (2.6) to the equation

$$16\Lambda^{\frac{3}{2}}\left[(Y^2-1)\frac{\partial T}{\partial X}-\left\{\frac{d}{dX}\ln(\mu_w\Lambda)\right\}(\tfrac{1}{3}Y^3-Y)\frac{\partial T}{\partial Y}\right]=\frac{\partial^2 T}{\partial Y^2}, \tag{2.12}$$

where
$$\Lambda = \rho_w \tau_w^3/4\mu_w^2(dp/dx)^2 \tag{2.13}$$

is a dimensionless function of X.

Equation (2.12) is intractable as it stands, but progress can be made for flows in which the second term in the square bracket can be discarded in comparison with the first. This will be the case (i) for a general Couette–Poiseuille channel flow, in which τ_w and dp/dx are constant (in this case, too, the quadratic used for u is exact;) (ii) for boundary-layer flows in which the length scale l say for heat addition is small compared with that of the surface over which the boundary layer has developed.

To be more precise, we require

$$\frac{\partial T}{\partial X}\bigg/\left(Y\frac{\partial T}{\partial Y}\right) \gg \frac{d}{dX}(\ln\Lambda)$$

for the term to be negligible, or, what is the same thing,

$$l^{-1} \gg (d/dx)(\ln\Lambda).$$

For example for a Falkner–Skan flow with external velocity proportional to x^m, $(d/dx)(\ln\Lambda) = (m+1)/2x$. We therefore require $x \gg l$ for such a flow, and the approximation would be inapplicable close to the leading edge. (The case of flow on a flat plate with zero pressure gradient is a degenerate one with $\Lambda^{-1} = 0$. In this case $(d/dx)(\ln\Lambda) = 0$ and the term does not appear: we then have the equation investigated by Lighthill, and his solution is readily recovered by an asymptotic method in §4). In fact, the analysis has been carried out with the experimental application in mind of measuring heat transfer from a gauge typically 1 mm long mounted on the surface of a plate perhaps 1 m from the leading edge: for this application there is no doubt about the validity of omitting the term.

Accordingly, we replace (2.12) by

$$16\Lambda^{\frac{3}{2}}(Y^2-1)\,\partial T/\partial X = \partial^2 T/\partial Y^2, \tag{2.14}$$

the solution of which will now be found by use of the Laplace transform. We set

$$\int_0^\infty e^{-sX}T(X,Y)\,dX = \overline{T}(s,Y), \tag{2.15}$$

taking $X = 0$ as the leading edge of the surface, or some other point upstream of the heated element. Since the equation is parabolic, the influence of the heating on the temperature field is felt only in the downstream direction, and the gradient of the additional temperature due to the element is zero on the line $X = 0$. The Laplace transform of $\partial T/\partial X$ is therefore just $s\overline{T}$. (This would not be true if the term $\partial^2 T/\partial X^2$ had been retained, and we should have had to use the Fourier transform.)

If the further transformations

$$Y = \frac{z}{2k^{\frac{1}{2}}}, \quad \text{where} \quad k = s^{\frac{1}{2}}\Lambda^{\frac{3}{4}} \tag{2.16}$$

and

$$\bar{T}(s, Y) = \bar{T}_w(s) f(z) \tag{2.17}$$

are made, then (2.14) is replaced by the hypergeometric equation

$$\frac{d^2 f}{dz^2} + (4k - z^2) f = 0 \tag{2.18}$$

with boundary conditions $f(2\sqrt{k}) = 1, \quad f(\infty) = 0.$ (2.19)

The solution is

$$f(z) = \frac{D_{2k-\frac{1}{2}}(z\sqrt{2})}{D_{2k-\frac{1}{2}}(2\sqrt{(2k)})}, \tag{2.20}$$

where D_ν is the parabolic cylinder function, whose properties are listed for instance by Erdelyi (1953, pp. 115 *et seq.*)

The expansion in ascending powers of z is

$$f(z) = \frac{\pi^{\frac{1}{2}} A(k)}{(-\frac{1}{4}-k)!}\left[1 - 2kz^2 + \ldots - \alpha(k)\left(z - \frac{2k}{3} z^3 + \ldots\right)\right], \tag{2.21}$$

where

$$A(k) = 2^{k-\frac{1}{4}}/D_{2k-\frac{1}{2}}(2\sqrt{[2k]})$$

and

$$\alpha(k) = 2(-\tfrac{1}{4}-k)!/(-\tfrac{3}{4}-k)! = \frac{(-\frac{1}{4})!}{2(\frac{1}{4})!}(1 - \pi k + O(k^2)).$$

The solution can also be written as a contour integral

$$f(z) = -\frac{(k-\frac{3}{4})!A(k)}{2\pi i}\frac{z}{\sqrt{2}}\int_{\infty}^{(1+)} \exp\left(-\tfrac{1}{2}z^2 w + k\ln\frac{1+w}{1-w}\right)\frac{dw}{(1-w^2)^{\frac{1}{4}}}, \tag{2.22}$$

3. Heat transfer to the wall

The heat transfer to the wall is given by

$$q_w(x) = \left(\lambda\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{2}\left(\frac{\lambda_w}{\mu_w}\right)\left(\frac{\rho_w \tau_w}{\Lambda}\right)^{\frac{1}{2}}\left(\frac{\partial T}{\partial Y}\right)_{Y=1} \tag{3.1}$$

The factor (λ_w/μ_w), being equal to $\sigma^{-1}C_p$, is independent of x according to our assumption in the previous section. The Laplace transform of $q_w/(\rho_w \tau_w)^{\frac{1}{2}}$ with regard to X can therefore be written

$$\mathscr{L}\frac{q_w}{(\rho_w\tau_w)^{\frac{1}{2}}} = \left(\frac{\lambda_w}{2\mu_w}\right)\Lambda^{-\frac{1}{2}}\left(\frac{\partial\bar{T}}{\partial Y}\right)_{Y=1} = \frac{\lambda_w}{2\mu_w}\Lambda^{-\frac{1}{2}}\bar{T}_w(s)\, 2\sqrt{k} f'(2\sqrt{k})$$

(where $k = s^{\frac{1}{2}}\Lambda^{\frac{3}{4}}$)

$$= -\frac{1}{2}\left(\frac{\lambda_w}{\mu_w}\right) s\bar{T}_w(s)\Phi(s) \tag{3.2}$$

say, where

$$\left.\begin{aligned} \Phi(s) &= (s\Lambda^{\frac{1}{2}})^{-1}\phi(s^{\frac{1}{2}}\Lambda^{\frac{3}{4}}),\\ \phi(k) &= -2\sqrt{k} f'(2\sqrt{k}), \end{aligned}\right\} \tag{3.3}$$

and $f(z)$ is the solution of (2.18). By the convolution theorem for Laplace transforms, we can then write q_w as a Stieltjes integral

$$q_w(x) = -\tfrac{1}{2}\lambda_w \left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} \int_{X=0}^{X(x)} F(X-S)\, dT_w(S), \tag{3.4}$$

where
$$F(X) = \mathscr{L}^{-1}\Phi(s), \tag{3.5}$$

$\mathscr{L}^{-1}$ being used to denote the inverse Laplace transform. In principle this completes the solution, since $\phi(k)$ can be calculated from tables of the known function

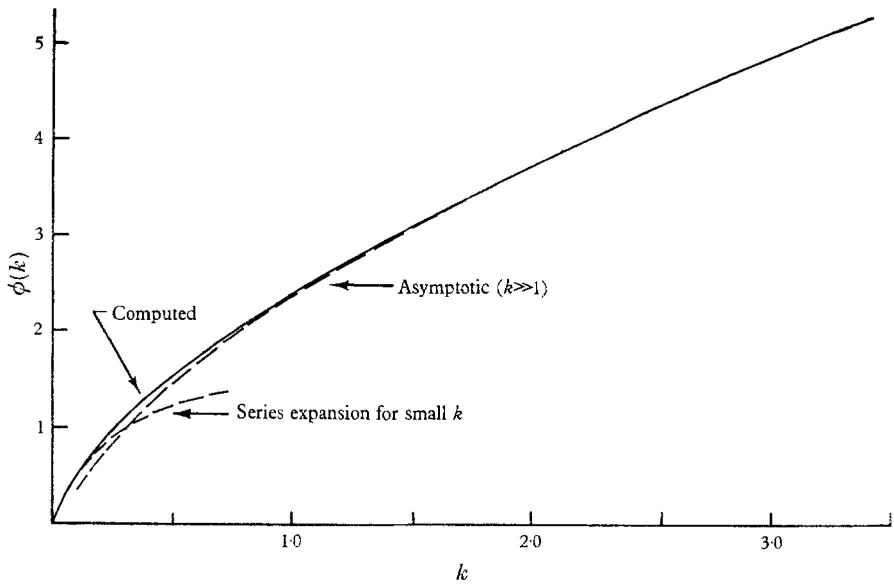

FIGURE 1. The function $\phi(k)$, equation (3.3).

$f(z)$ and the inversion (3.5) could be carried out numerically for any given value of Λ. $\phi(k)$ is plotted in figure 1. Useful limiting results can however be obtained analytically for the cases when Λ is very large or very small compared with unity, which correspond respectively to flows with small pressure gradient and finite skin friction (near flat-plate flow) and finite presure gradient and small skin friction (near separated flow). These are obtained by examining the asymptotic behaviour of $\phi(k)$ for large and small k respectively, before performing the inversion.

3.1. *Near flat plate flow*, $\Lambda \gg 1$

The asymptotic behaviour of $\phi(k)$ for $k \gg 1$ is most easily found from the contour integral representation of $f(z)$, equation (2.22). If we set $z = 2\sqrt{k}$ in this equation, and define

$$I_n(k) = \int_{\infty}^{(1+)} w^n (1-w^2)^{-\frac{1}{4}} e^{kg(w)} dw \tag{3.6}$$

for $n = 0, 1$, where

$$g(w) = -2w + \ln\frac{1+w}{1-w} = 2\left(\frac{w^3}{3} + \frac{w^5}{5} + \ldots\right), \tag{3.7}$$

then (3.3) is
$$\phi(k) = \frac{4kI_1(k)}{I_0(k)} - 1. \tag{3.8}$$

The integrals I_0 and I_1 can be estimated for large k by the method of steepest descents. The saddle point is at $w = 0$, and since g behaves like w^3 here, there are, in contrast to the usual situation, *three* hills and *three* valleys, as indicated in

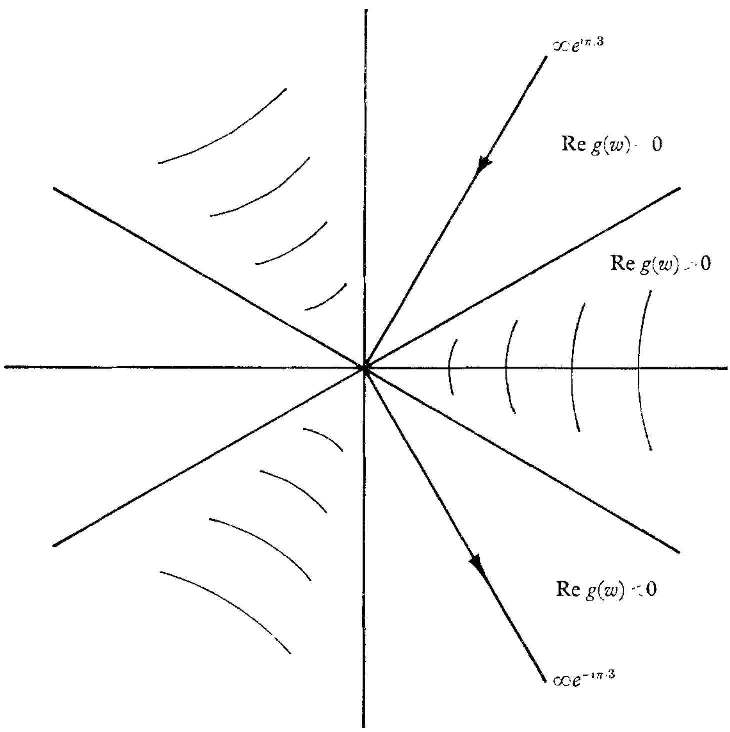

FIGURE 2. Steepest descent paths for evaluating I_0 and I_1.

figure 2. The path of integration in each integral can be deformed to the two straight lines from $\infty e^{\frac{1}{3}i\pi}$ to 0 and from 0 to $\infty e^{-\frac{1}{3}i\pi}$, which are the steepest descent paths at 0. If we set $w = \epsilon_1 u$ on the first, and $w = \epsilon_2 u$ on the second, where

$$\epsilon_1 = k^{-\frac{1}{3}} e^{\frac{1}{3}i\pi}, \quad \epsilon_2 = k^{-\frac{1}{3}} e^{-\frac{1}{3}i\pi},$$

we obtain

$$I_0 = \epsilon_1 \int_\infty^0 (1-\epsilon_1^2 u^2)^{-\frac{1}{4}} \exp\left(-\tfrac{2}{3}u^3 - \tfrac{2}{5}\epsilon_1^2 u^5 - \ldots\right) du + \epsilon_2 \int_0^\infty (1-\epsilon_2^2 u^2)^{-\frac{1}{4}} \exp\left(-\tfrac{2}{3}u^3 - \tfrac{2}{5}\epsilon_2^2 u^5 - \ldots\right) du.$$

When the integrands are expanded in powers of the ϵ's and combined the result is

$$\begin{aligned} I_0 &= -i\sqrt{3}\,k^{-\frac{1}{3}} \int_0^\infty e^{-\frac{2}{3}u^3}\{1 - k^{-\frac{4}{3}}(\tfrac{5}{32}u^4 - \tfrac{27}{70}u^7 + \tfrac{2}{25}u^{10}) + O(k^{-2})\}\,du \\ &= -i\sqrt{3}\,k^{-\frac{1}{3}}[(\tfrac{3}{2})^{-\frac{2}{3}}(-\tfrac{2}{3})! + \tfrac{9}{1120}(\tfrac{3}{2})^{\frac{2}{3}}(\tfrac{2}{3})!\,k^{-\frac{4}{3}} + O(k^{-2})]. \end{aligned} \tag{3.9}$$

In exactly the same way we find

$$I_1 = -\tfrac{1}{2}i\sqrt{3}\,k^{-\frac{2}{3}}[(\tfrac{3}{2})^{-\frac{1}{3}}(-\tfrac{1}{3})!+\tfrac{11}{20}(\tfrac{3}{2})^{\frac{1}{3}}(\tfrac{1}{3})!k^{-\frac{2}{3}}+O(k^{-2})]. \tag{3.10}$$

Insertion of these results in (3.8) gives

$$\phi(k) = \sum_{n=0}^{\infty} a_n k^{\frac{2}{3}(1-n)},$$

where
$$a_0 = 4(\tfrac{3}{2})^{\frac{1}{3}}\frac{(-\frac{1}{3})!}{(-\frac{2}{3})!}, \quad a_1 = \tfrac{1}{10}, \quad a_2 = \frac{9a_0^2}{4480}, \ldots \tag{3.11}$$

Hence
$$\Phi(s) = \sum_{n=0}^{\infty} a_n \Lambda^{-\frac{1}{2}n} s^{-\frac{1}{3}(2+n)} \tag{3.12}$$

and Laplace inversion gives

$$F(X) = \sum_{n=0}^{\infty} \frac{a_n \Lambda^{-\frac{1}{2}n}}{(\frac{1}{3}(n-1))!} X^{\frac{1}{3}(n-1)} \tag{3.13}$$

as the expansion appropriate for large values of Λ, to be inserted in (3.4). This leads to an expansion of q_w in powers of $\Lambda^{-\frac{1}{2}}\sigma^{-\frac{1}{3}}$ of the form

$$q_w(x) = -\lambda_w \left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} \sigma^{\frac{1}{3}} \sum_{n=0}^{\infty} \frac{A_n}{(\Lambda^{\frac{1}{2}}\sigma^{\frac{1}{3}})^n} \int_0^x \left[\int_s^x \left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} d\xi\right]^{\frac{1}{3}(n-1)} dT_w(s), \tag{3.14}$$

where
$$A_n = 2^{\frac{1}{3}(2n-5)}/a_n\,(\tfrac{1}{3}n-\tfrac{1}{3})!.$$

In the limit $\Lambda \to \infty$, which corresponds to $dp/dx = 0$, $\tau_w \neq 0$, only the first term is present; this is precisely that given by Lighthill (1950, equation (29)), appropriately generalized to compressible flow, the coefficient being

$$A_0 = [3^{\frac{2}{3}}(\tfrac{1}{3})!]^{-1} = 0{\cdot}539.$$

Liepmann's (1958) value for the constant was 0·524. This limit may also be regarded as that of high Prandtl number, its physical significance being that heat does not penetrate beyond the linear part of the velocity profile if the conductivity is small enough.

3.2. *Flow close to a separation point*

Liepmann (1958) assumed that the flow near a separation point could be represented by a quadratic profile of the form (2.8) with $\tau_w = 0$, $dp/dx > 0$. This corresponds to $\Lambda = 0$, and we now show that Liepmann's semi-empirical treatment of this limit can be put on the same theoretical footing as that for the opposite limit. Accordingly, we examine the solution when Λ is small compared with unity. It must be recognized however that $\ln\Lambda$ would vary rapidly close to a separation point x_s where, according to boundary-layer theory, $\tau_w \propto |x-x_s|^{\frac{1}{2}}$, so if dp/dx remains constant, the limit $\tau_w = 0$ is an unrealistic representation of the flow at any appreciable distance from x_s; in fact since Λ is proportional to (Reynolds number)$^{\frac{1}{2}}$, the large Λ discussion is the physically significant one until one is very close indeed to x_s. But we include for completeness a discussion for $\Lambda \ll 1$.

Then $k \ll 1$, by (2.16), and use of (2.21) in (3.3) shows that

$$\phi(k) = \sum_{n=0}^{\infty} b_n k^{\frac{1}{2}(n+1)}, \quad \text{i.e.} \quad \Phi(s) = \sum_{n=0}^{\infty} b_n s^{\frac{1}{4}(n-3)} \Lambda^{\frac{1}{8}(3n-1)}, \tag{3.15}$$

where $\qquad b_0 = (-\tfrac{1}{4})!/(\tfrac{1}{4})!, \quad b_1 = b_0^2, \quad b_2 = b_0^3 - \pi b_0, \quad \text{etc.},$

and Laplace inversion gives the asymptotic expansion

$$F(X) \sim \sum_{n=0}^{\infty} \frac{b_n}{(-\frac{1}{4}(n+1))!} \Lambda^{\frac{1}{8}(3n-1)} X^{-\frac{1}{4}(n+1)} \tag{3.16}$$

for large X. In obtaining an expression for heat transfer in this case it is better to eliminate τ_w in favour of dp/dx from the expressions for X and q_w, by means of

$$\left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} = (4\Lambda)^{\frac{1}{6}} \left(\frac{\rho_w}{\mu_w^2}\frac{dp}{dx}\right)^{\frac{1}{3}}. \tag{2.13}$$

With this modification, insertion of (3.16) in (3.4) provides an asymptotic expansion in powers of $\Lambda^{\frac{1}{3}}\sigma^{\frac{1}{4}}$ of the form

$$q_w(x) = -\lambda_w \left(\frac{p_w}{\mu_w^2}\frac{dp}{dx}\right)^{\frac{1}{3}} \sigma^{\frac{1}{4}} \sum_{n=0}^{\infty} B_n (\Lambda^{\frac{1}{3}}\sigma^{\frac{1}{4}})^n \int_0^x \left[\int_s^x \left(\frac{\rho_w}{\mu_w^2}\frac{dp}{d\xi}\right)^{\frac{1}{3}} d\xi\right]^{-\frac{1}{4}(n+1)} dT_w(s), \tag{3.17}$$

where $\qquad B_n = b_n[2^{\frac{5}{4}+\frac{7}{12}n}(-\tfrac{1}{4}n - \tfrac{1}{4})!]^{-1}.$

For a *stepwise* distribution of temperature, the integrals on the right are divergent for $n > 2$, and it is necessary to take more careful account of the behaviour of $F(X)$ near $X = 0$, which is not described by the asymptotic series (3.16). This is considered in §4.4.

The limit $\Lambda = 0$ corresponds to the case investigated by Liepmann, whose expression for a stepwise distibution of wall temperature is then identical in form with the leading term of (3.16). Our constant B_0 is $[2^{\frac{5}{4}}(\frac{1}{4})!]^{-1} = 0{\cdot}4639$, whereas Liepmann's corresponding constant $(3\beta/8)^{\frac{1}{4}}$ has the value 0·4594 when β is estimated in the manner he suggests. One exact solution of the full boundary-layer equations exists for heat transfer at a separation point in Falkner–Skan flow, with $m = -0{\cdot}0904$ at constant wall temperature T_w. This leads to an expression

$$q_w = -\lambda_w f(\sigma) T_w,$$

where for large σ, $f(\sigma) \sim B_0 \sigma^{\frac{1}{4}}$, in perfect agreement with the above limiting result. For $\sigma = 0{\cdot}7$, however, the value of $Nu/(Re)^{\frac{1}{2}}$ for the same flow would be 0·438, whereas (3.17) with $\Lambda = 0$ would give 0·453. Liepmann's calculation gave 0·448.

4. Nusselt number for an isolated surface element

The foregoing theory has been developed with the experimental application in mind of measuring skin friction by means of a surface film of small streamwise extent l, say, heated to a temperature T_0 above that of the surrounding wall. We represent this by the 'top-hat' surface temperature distribution

$$T_w(x) = \begin{matrix} 0 \\ T_0 \end{matrix} \quad |x - x_0| \gtrless \tfrac{1}{2}l. \tag{4.1}$$

If the gauge is sufficiently small, we may treat μ_w, ρ_w and dp/dx as constant over the length of surface over which the temperature field is significantly modified by its presence, and use the results of the last section with the appropriate value of Λ. Dimensional analysis then shows that if a Nusselt number for the gauge is defined by

$$Nu = (\lambda_w T_0)^{-1} \int_{x_0-\frac{1}{2}l}^{x_0+\frac{1}{2}l} q_w(x)\, dx, \tag{4.2}$$

Nu can depend only on the two parameters

$$\alpha = \frac{\sigma \rho_w \tau_w l^2}{\mu_w^2}, \quad \beta = \frac{\sigma \rho_w}{\mu_w^2}\left(\frac{dp}{dx}\right) l^3, \tag{4.3}$$

and must be of the form $\alpha^{\frac{1}{3}} f(\beta/\alpha^{\frac{3}{2}})$ if the energy equation is in the form (2.3). These parameters are related to Λ by

$$\alpha^3/\beta^2 = 4\sigma\Lambda, \tag{4.4}$$

and we may also note that the ratio of the pressure drop over the length of the element to the shear force on it, a natural parameter in which to expand the Nusselt number, is

$$\frac{\beta}{\alpha} = \frac{l}{\tau_w}\frac{dp}{dx}. \tag{4.5}$$

We now express Nu in terms of α and β.

One caveat must be entered at this stage concerning the applicability of the energy equation in the form we have used up to now: should we retain the term $\partial/\partial x(\lambda[\partial T/\partial x])$ on the right-hand side of (2.2) when there is an abrupt discontinuity in $T_w(x)$? To do so accurately makes the analysis of §§2 and 3 a good deal more elaborate; but it is easy simply to estimate the order of magnitude of the influence of the extra term by a dimensional analysis of the full energy equation, which is carried out in §4.4. In fact, a typical experimental value for α is of the order of 50, which means that the gauge is long enough for the effect to be unimportant; but before looking more closely into this we obtain the necessary results with the term left out.

4.1. *Near-flat-plate flow*

Since the solution of the energy equation has been carried out in co-ordinates transformed in accordance with (2.11), we take the boundary conditions (4.1) as applying in the transformed intervals $|X - X_0| \lessgtr \frac{1}{2}L$, say, where

$$L = \frac{4}{\sigma}\left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} l = 4\left(\frac{\alpha}{\sigma^3}\right)^{\frac{1}{2}}. \tag{4.6}$$

Then substitution of (4.1) in (3.4) shows that for $X_0 - \frac{1}{2}L < X < X_0 + \frac{1}{2}L$,

$$q_w(x) = -\tfrac{1}{2}\lambda_w T_0 \left(\frac{\rho_w \tau_w}{\mu_w^2}\right)^{\frac{1}{2}} F(X - X_0 + \tfrac{1}{2}L). \tag{4.7}$$

Integration with respect to x gives

$$Nu = \tfrac{1}{8}\sigma \int_0^L F(X')\, dX', \tag{4.8}$$

where we have written $X-(X_0-\frac{1}{2}L) = X'$, and used the fact that

$$dx = \frac{\sigma\mu_w}{4(\rho_w\tau_w)^{\frac{1}{2}}}dX.$$

For near-flat-plate flow, $F(X)$ is given by (3.13), and on integration

$$Nu = \tfrac{1}{8}\sigma L^{\frac{2}{3}} \sum_{n=0}^{\infty} \frac{a_n}{(\frac{1}{3}n+\frac{2}{3})!}\Lambda^{-\frac{1}{3}n}L^{\frac{1}{3}n}. \tag{4.9}$$

Expressed in terms of α and β defined by (4.3), this is

$$Nu = \alpha^{\frac{1}{3}} \sum_{n=0}^{\infty} C_n \left(\frac{\beta}{\alpha^{\frac{1}{3}}}\right)^n, \tag{4.10}$$

where $$C_n = 2^{\frac{5}{3}(n-1)}\,a_n/(\tfrac{1}{3}n+\tfrac{2}{3})!.$$

The first three terms are

$$Nu = 0{\cdot}8072\left(\frac{\sigma\rho_w\tau_w l^2}{\mu_w^2}\right)^{\frac{1}{3}} + 0{\cdot}10\frac{l}{\tau_w}\frac{dp}{dx} - 0{\cdot}0287\left(\frac{\sigma\rho_w\tau_w l^2}{\mu_w^2}\right)^{-\frac{1}{3}}\left(\frac{l}{\tau_w}\frac{dp}{dx}\right)^2 + \dots. \tag{4.11}$$

4.2. *Flow near a separation point*

In this case too the Nusselt number is given by the integral (4.8), but a difficulty arises when the expansion (3.16) for $F(X)$ is inserted, since the integrals of all the terms beyond the third diverge at the lower limit. However we can split up the integral as follows:

$$\int_0^L F(X)\,dX = \int_0^\infty \left\{F(X) - \sum_{n=0}^{2} b'_n X^{-\frac{1}{4}(n+1)}\right\}dX + \int_0^L \left(\sum_{n=0}^{2} b'_n X^{-\frac{1}{4}(n+1)}\right)dX - \int_L^\infty \left\{F(X) - \sum_{n=0}^{2} b'_n X^{-\frac{1}{4}(n+1)}\right\}dX, \tag{4.12}$$

where b'_n is written for $b_n\Lambda^{\frac{1}{8}(3n-1)}/(-\frac{1}{4}n-\frac{1}{4})!$. It will be noted that

$$b'_3 = b'_7 = \dots = 0.$$

The second integral can be evaluated as it stands. In the third, we can replace the integrand by the asymptotic expansion

$$\sum_{n=4}^{\infty} b'_n X^{-\frac{1}{4}(n+1)},$$

provided L (or more strictly $\Lambda^{\frac{3}{8}}L^{\frac{1}{4}}$) is large enough. It remains to evaluate the first term. This can be looked on as the limit as $s\to 0$ of

$$\int_0^\infty \left\{F(X) - \sum_{n=0}^{2} b'_n X^{-\frac{1}{4}(n+1)}\right\}e^{-sX}\,dX$$

i.e. of $$\Phi(s) - \sum_{n=0}^{2} b_n \Lambda^{\frac{1}{8}(3n-1)} s^{\frac{1}{4}(n-3)},$$

and this limit, by (3.15), is just $b_3\Lambda$. Altogether we find therefore, that for large $\Lambda^{\frac{3}{8}}L^{\frac{1}{4}}$,

$$\int_0^L F(X)\,dX \sim \sum_{n=0}^{\infty} \frac{b_n\Lambda^{\frac{1}{8}(3n-1)}}{(\frac{3}{4}-\frac{1}{4}n)!} L^{\frac{1}{4}(3-n)} \tag{4.13}$$

(the terms for $n = 7, 11, \ldots$ are now absent, but *not* that for $n = 3$). Substitution for Λ and L from (4.4) and (4.6) gives

$$Nu = \beta^{\frac{1}{4}} \sum_{n=0}^{\infty} D_n \left(\frac{\alpha}{\beta^{\frac{3}{4}}}\right)^n, \tag{4.14}$$

where
$$D_n = 2^{-\frac{5}{4}(n+1)}\, b_n / (\tfrac{3}{4} - \tfrac{1}{4}n)!.$$

The first three terms are

$$Nu = 0{\cdot}6184 \left(\frac{\sigma\rho_w l^3}{\mu_w^2}\frac{dp}{dx}\right)^{\frac{1}{4}} + 0{\cdot}3649 \left(\frac{l}{\tau_w}\frac{dp}{dx}\right)^{-1} \left(\frac{\sigma\rho_w l^3}{\mu_w^2}\frac{dp}{dx}\right)^{\frac{1}{2}} - 0{\cdot}1457 \left(\frac{l}{\tau_w}\frac{dp}{dx}\right)^{-2} \left(\frac{\sigma\rho_w l^3}{\mu_w^2}\frac{dp}{dx}\right)^{\frac{3}{4}}. \tag{4.15}$$

4.3. *Comparison with an integral solution*

From the thermal energy integral equation Curle (1961) derives a differential equation for the heat transfer from a surface to a quadratic shear profile. For constant skin friction and pressure gradient and a 'top-hat' temperature distribution Curle's equation may be immediately integrated to give, in our notation,

$$\tfrac{2}{3}a\alpha - \frac{3b}{8}\beta\frac{kT_0}{lq_w(x)} = -\frac{xl^2q_w^3(x)}{k^3T_0^3} \quad \text{for} \quad |x-x_0| < \tfrac{1}{2}l, \tag{4.16}$$

where a and b are the constants given by Curle. In order to make a comparison with our results $q_w(x)$ is found as an expansion for small and large $\beta/\alpha^{\frac{4}{3}}$ and then integrated over the length l to give

$$\beta/\alpha^{\frac{4}{3}} \ll 1: \quad Nu = (\tfrac{9}{4}a\alpha)^{\frac{1}{3}}\left[1 + \tfrac{3}{16}(\tfrac{4}{9})^{\frac{1}{3}}\frac{b\beta}{(a\alpha)^{\frac{4}{3}}} + \ldots\right], \tag{4.17}$$

$$\beta/\alpha^{\frac{4}{3}} \gg 1: \quad Nu = (\tfrac{32}{27}b\beta)^{\frac{1}{4}}\left[1 + \tfrac{2}{3}(\tfrac{3}{8})^{\frac{1}{4}}\frac{a\alpha}{(b\beta)^{\frac{3}{4}}} + \ldots\right]. \tag{4.18}$$

These are identical in form with (4.10) and (4.14) and the leading terms differ by only $1\frac{1}{2}$ and 4 % respectively. It is not surprising that the leading terms agree so well because Curle has chosen a and b such that the prediction of heat transfer by his method agrees closely (1 %) with the exact solution for the heat transfer to the Falkner–Skan similarity profiles, and the limits $\beta = 0$ and $\alpha = 0$ correspond to the zero pressure gradient and zero skin friction similarity profiles (particularly for large Prandtl number). The coefficients of the second terms are in error by 12 % and 3 % compared with those of (4.10) and (4.14). This rather close agreement, however, is further support for the accuracy of Curle's method for the prediction of heat transfer.

It is worth considering the approximate relationship between Nu, α and β which may be obtained from Curle's differential equation if it is assumed that $q_w(x)$ is proportional to a function of x which does not depend on skin friction or pressure gradient. In this case (4.16) leads to the simple result

$$c_1\alpha + c_2\frac{\beta}{Nu} = Nu^3, \tag{4.19}$$

where the constants c_1 and c_2 depend on the assumed function of x. The functional form of this equation

$$Nu = \alpha^{\frac{1}{3}} f\left(\frac{\beta}{\alpha^{\frac{4}{3}}}\right)$$

is of the same form as our results and, indeed, if c_1 and c_2 have the respective values 0·5266 and 0·1463 (chosen so that (4.19) agrees exactly with (4.10) and (4.14) in the two limits $\beta = 0$, $\alpha = 0$) then (4.19) becomes an equation for Nu that

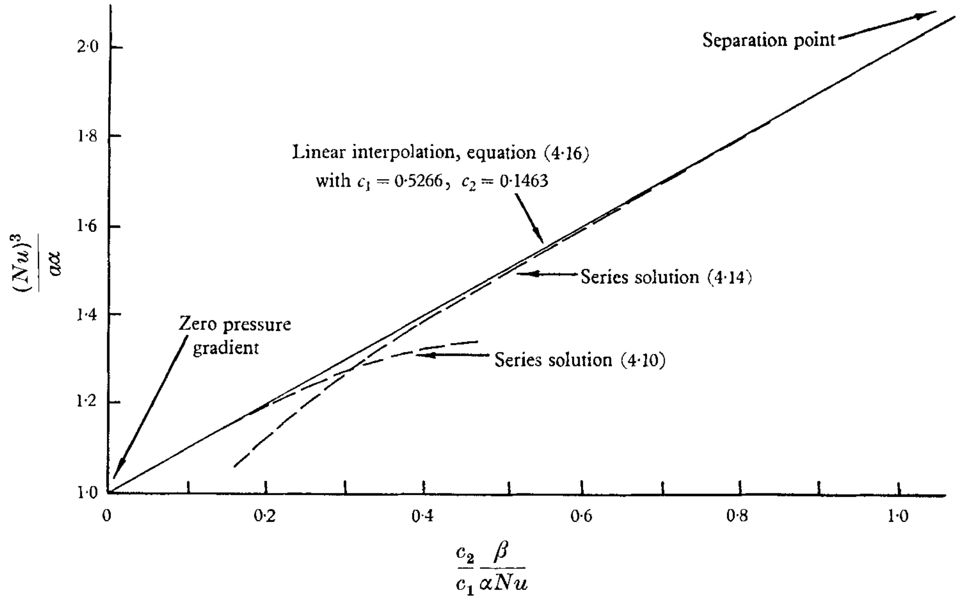

FIGURE 3. Linear interpolation to asymptotic formulae. The relation between α, β and Nu given by the interpolated line has a form suggested by a consideration of the integral approximation of Curle (1962).

by good fortune agrees closely with our results over the whole range of $\beta/\alpha^{\frac{4}{3}}$, as shown in figure 3. As a result this simpler relationship may be used for experimental purposes instead of the two series.

We note that these values of c_1 and c_2 may be compared with those obtained by assuming a function for $q_w(x)$. Bellhouse & Schultz (1966) treated $q_w(x)$ as constant over the length of the gauge and obtained (4.19), their equation (A 1.2), with c_1 and c_2 0·2226 and 0·0523 respectively. A more accurate result may be obtained if $q_w(x)$ is assumed proportional to x^n. Liepmann & Skinner (1954) showed that for zero-pressure gradient n is $-\frac{1}{3}$ and for zero skin friction finite pressure gradient n is $-\frac{1}{4}$. If n is taken as $-\frac{1}{3}$ independent of pressure gradient c_1 and c_2 have the values 0·501 and 0·353.

4.4. *Effect of gauge length*

To investigate the way in which the above results would be modified if the term

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)$$

in (2.2) were retained, as would certainly be necessary if the gauge length l were sufficiently small, we may look at the full energy equation for a quadratic velocity profile independent of x, assuming constant fluid properties and making lengths non-dimensional with l by means of

$$\xi = x/l, \quad \eta = y/l.$$

Equation (2.2) is then

$$(\alpha\eta + \tfrac{1}{2}\beta\eta^2)\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2}, \tag{4.20}$$

where α and β are the parameters defined in (4.3), with boundary conditions

$$T(\xi, 0) = \begin{cases} 0 \\ 1 \end{cases} (|\xi - \xi_0| \gtrless \tfrac{1}{2}), \quad T(\xi, \infty) = 0.$$

The Nusselt number for the gauge is

$$Nu = \int_{\xi_0 - \frac{1}{2}}^{\xi_0 + \frac{1}{2}} \frac{\partial T}{\partial \eta}(\xi, 0)\, d\xi. \tag{4.21}$$

The boundary condition (4.20) ensures that variations in the ξ direction, represented by the terms $\partial T/\partial \xi$ and $\partial^2 T/\partial \xi^2$, are $O(1)$, and we now scale the η coordinate so as to make the coefficients of the first and third terms in the equation of the same order. Firstly, for near flat plate flow with $\beta/\alpha^{\frac{4}{3}}$ small, we write

$$\eta = \alpha^{-\frac{1}{3}}\eta',$$

when (4.19) becomes

$$\left(\eta' + \frac{1}{2}\frac{\beta}{\alpha^{\frac{4}{3}}}\eta'^2\right)\frac{\partial T}{\partial \xi} = \alpha^{-\frac{2}{3}}\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta'^2}. \tag{4.22}$$

Hence if $\alpha^{-\frac{2}{3}} \ll 1$, the term in $\partial^2 T/\partial \xi^2$ can be omitted; the Nusselt number is then exactly that found in §4.1. However, it is only if $\beta/\alpha^{\frac{4}{3}} \gg \alpha^{-\frac{2}{3}}$ that this expansion correctly represents the effect of the pressure gradient; if this inequality is not satisfied, we should also take account of gauge size.

To look at the opposite limit of flow near a separation point, when $\alpha/\beta^{\frac{3}{4}}$ is small, we write $\eta = \beta^{-\frac{1}{4}}\eta''$, when (4.19) becomes

$$\left(\frac{\alpha}{\beta^{\frac{3}{4}}}\eta'' + \tfrac{1}{2}\eta''^2\right)\frac{\partial T}{\partial \xi} = \beta^{-\frac{1}{2}}\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta''^2}. \tag{4.23}$$

Thus if the gauge is long enough for $\beta^{-\frac{1}{2}}$ to be small compared with unity, the expression for Nu found in §4.2 can be used, with the proviso that it correctly represents the influence of the parameter $(\tau_w/l)/(dp/dx)$ only if the further inequality $\alpha/\beta^{\frac{3}{4}} \gg \beta^{-\frac{1}{2}}$ is also satisfied. We have in fact solved (4.19) in the two extreme cases $\beta = 0$ (flat plate) and $\alpha = 0$ (separation point), and obtained the following expressions for Nusselt number:

flat plate:

$$Nu = \frac{\alpha^{\frac{1}{3}}\phi_0}{(\frac{2}{3})!} - \frac{\alpha^{-\frac{1}{3}}\phi_0^2}{(-\frac{2}{3})!} + O(\alpha^{-\frac{5}{3}}), \tag{4.24}$$

where

$$\phi_0 = -Ai'(0)/Ai(0) = 3^{\frac{1}{3}}(-\tfrac{1}{3})!/(-\tfrac{2}{3})!;$$

separation point:

$$Nu = (\tfrac{1}{2}\beta)^{\frac{1}{4}}(\tfrac{1}{2}b_0)\left[\frac{1}{(\frac{3}{4})!} - \tfrac{1}{4}\pi\frac{(\frac{1}{2}\beta)^{-\frac{1}{2}}}{(-\frac{3}{4})!} + O(\beta^{-1})\right], \tag{4.25}$$

where b_0 is the constant previously defined. The leading terms are identical with those of (4.10) and (4.14) respectively.

5. Heat flux across a plane parallel to the surface

For the foregoing theory to be applicable to skin friction measurement in a turbulent boundary layer, the velocity profile must be well represented by (1.1) within the region to which the heat from the gauge penetrates. According to the law of the wall, the sublayer is of this form, whatever the pressure gradient, up to the point where $yu_\tau/\nu = 12$, where u_τ is the friction velocity $(\tau_w/\rho)^{\frac{1}{2}}$. To assess the relevance of the theory, we examine in this section the flux of heat across a plane $y =$ constant, as a function of the parameter α, which can be rewritten

$$\alpha = \sigma\left(\frac{u_\tau l}{\nu}\right)^2. \tag{5.1}$$

We assume that the thermal conductivity within the sublayer is still molecular, despite the possible presence of turbulent eddies. If the velocity profile is independent of pressure gradient, and the condition $\alpha^{\frac{2}{3}} \gg 1$ derived in §4.4 is satisfied, the energy equation may then be taken as

$$\alpha\eta\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \eta^2}, \tag{5.2}$$

where $\xi = (x - x_0 + \frac{1}{2}l)/l$, $\eta = y/l$ (so that the heated element extends from $\xi = 0$ to $\xi = 1$). On Laplace transformation with respect to ξ, this takes the form of Airy's equation

$$\frac{d^2\overline{T}}{dz^2} = z\overline{T}, \tag{5.3}$$

where
$$\overline{T} = \int_0^\infty e^{-s\xi}T(\xi,\eta)\,d\xi, \quad z = (\alpha s)^{\frac{1}{3}}\eta. \tag{5.4}$$

The solution is
$$\overline{T} = \overline{T}_w(s)\frac{Ai(z)}{Ai(0)}, \tag{5.5}$$

where $Ai(z)$ is the Airy function (Jeffreys & Jeffreys 1946, p. 477). It is sufficient to calculate the heat transfer for the case when $T_w(\xi)$ is a step function at $\xi = 0$, and subsequently to find that for a top hat distribution by subtracting from it the solution for an equal step function at $\xi = 1$. Accordingly we set $\overline{T}_w(s) = \Delta T/s$ in (5.5).

The flux of heat outwards across unit area of the plane $\eta =$ constant is

$$-\frac{\lambda}{l}\frac{\partial T}{\partial \eta},$$

with Laplace transform
$$-\frac{\lambda \Delta T}{l}\frac{\alpha^{\frac{1}{3}}}{s^{\frac{2}{3}}}Ai'[(\alpha s)^{\frac{1}{3}}\eta]/Ai(0),$$

which on inversion gives the heat flux as

$$q(\xi,\eta) = \frac{\lambda\Delta T}{l\eta}Q\left(\frac{\xi}{\alpha\eta^3}\right) \tag{5.6}$$

say, where
$$Q(\gamma) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Ai'(t^{\frac{1}{3}})}{Ai(0)}e^{\gamma t}\frac{dt}{t^{\frac{2}{3}}}. \tag{5.7}$$

The heat flux from a top hat distribution in T_w is given by (5.6) for $0 < \xi < 1$, and by

$$\frac{\lambda\Delta T}{l}\left[Q\left(\frac{\xi}{\alpha\eta^3}\right)-Q\left(\frac{\xi-1}{\alpha\eta^3}\right)\right] \tag{5.8}$$

for $\xi > 1$.

For large γ,
$$Q(\gamma) \sim \frac{3^{\frac{1}{3}}}{(-\frac{2}{3})!}\gamma^{-\frac{1}{3}}. \tag{5.9}$$

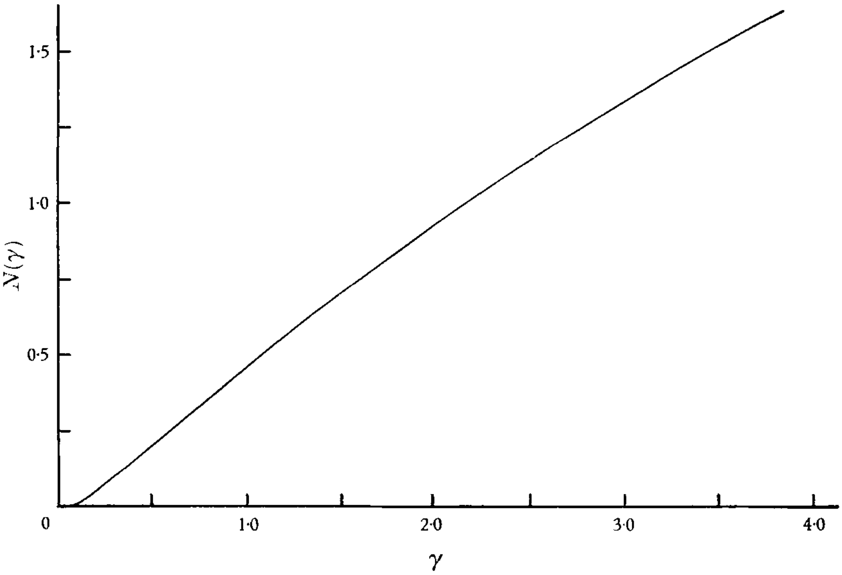

FIGURE 4. The function $N(\gamma)$, equation (5.11).

The Nusselt number for heat flux across a length $l\xi$ of the plane η = constant is

$$Nu(\xi,\eta) = \frac{l}{\lambda\Delta T}\int_0^{\xi} q(\xi,\eta)\,d\xi = \alpha\eta^2 N\left(\frac{\xi}{\alpha\eta^3}\right) \tag{5.10}$$

say, for $0 < \xi < 1$, where
$$N(\gamma) = \int_0^{\gamma} Q(\gamma')\,d\gamma'; \tag{5.11}$$

for $\xi > 1$ it is
$$\alpha\eta^2\left[N\left(\frac{\xi}{\alpha\eta^3}\right)-N\left(\frac{\xi-1}{\alpha\eta^3}\right)\right]. \tag{5.12}$$

The quantity $N(\gamma)$ has been computed from expansions of (5.7) for small and large γ and is plotted in figure 4. As $\gamma \to \infty$

$$N(\gamma) \sim [3^{\frac{1}{3}}/2(\tfrac{1}{3})!]\gamma^{\frac{2}{3}} = K\gamma^{\frac{2}{3}} \quad \text{say.}$$

This provides a check that the Nusselt number of the gauge itself, given by setting $\xi = 1$, $\eta = 0$ in (5.10) is
$$(3\alpha)^{\frac{1}{3}}/2(\tfrac{1}{3})!. \tag{5.13}$$

We also see that for *all* η, the limit as $\xi \to \infty$ of (5.12) is zero, since

$$Nu(\xi,\eta) \sim K\alpha^{\frac{1}{3}}[\xi^{\frac{2}{3}}-(\xi-1)^{\frac{2}{3}}] \to 0 \quad \text{as} \quad \xi \to \infty;$$

thus the total amount of heat crossing any plane η = constant (including the plane $\eta = 0$) is zero.

Because of the parabolic form of the energy equation, the heat flux from the element $0 < \xi < 1$, $\eta = 0$, is governed only by the flow in the strip $0 < \xi < 1$. An indication of the applicability of these results is therefore given by a graph of the Nusselt number for length l of the plane $\eta =$ constant; if this has become very

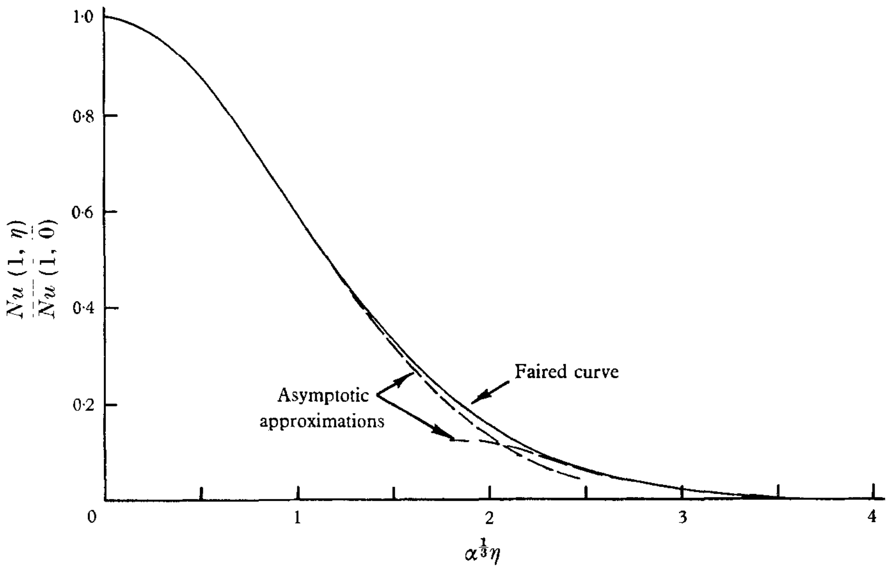

FIGURE 5. Ratio of heat crossing the part of the plane $y = \eta l$ directly above the gauge to total heat transferred from gauge to stream.

small compared with the Nusselt number at the wall for values of η lying within the sublayer, they may safely be used for inferring turbulent skin friction from the heat transfer by means of (5.13). The ratio is

$$\frac{Nu(1,\eta)}{Nu(1,0)} = \frac{2(\frac{1}{3})!(\alpha\eta^3)^{\frac{2}{3}}}{3^{\frac{1}{3}}} N\left(\frac{1}{\alpha\eta^3}\right). \tag{5.14}$$

This is plotted as a function of $\alpha^{\frac{1}{3}}\eta$ in figure 5, and it is seen that less than 1 % of the heat supplied from the surface crosses the plane $\eta = 3\alpha^{-\frac{1}{3}}$.

At this plane we have

$$\frac{yu_\tau}{\nu} = 3\left(\frac{lu_\tau}{\nu}\right)\alpha^{-\frac{1}{3}} = 3\left(\frac{u_\tau l}{\nu\sigma}\right)^{\frac{1}{3}}.$$

This must be less than about 12 if it is to be within the sublayer, so we must have

$$\alpha^{\frac{2}{3}} = \sigma^{\frac{2}{3}}(u_\tau l/\nu)^{\frac{4}{3}} < 256\sigma^2$$

and since we also require $\alpha^{\frac{2}{3}} > 12{\cdot}3$ for the result (5.13) to give τ_w within 5 %, it appears that the gauge length must be such that the Reynolds number $u_\tau l/\nu$ lies in the range, say

$$6{\cdot}6/\sigma^{\frac{1}{2}} < u_\tau l/\nu < 64\sigma. \tag{5.15}$$

This is discussed in reference to the experiments in the next section.

6. Application of the results and some experiments

6.1. *Applicability of the analysis*

The heated film technique to measure skin friction has been used by Bellhouse & Schultz (1966) and Brown (1967*a*, *b*). A platinum film is maintained at constant temperature with the usual hot-wire, constant resistance equipment and, if the losses to the substrate are constant, the additional power supplied to the element is related to skin friction, pressure gradient and fluid properties by the relations given in §4 provided: (1) $\alpha^{\frac{2}{3}} \gg 1$ so that the 'boundary layer' form of the energy equation applies; (2) the length of the film is sufficiently small for the penetration of heat to be confined to the part of the velocity profile that is adequately described by the first two terms; (3) the approximation of a top hat temperature distribution is sufficiently accurate.

Condition (1) has been described in §§4.4 and 5 and leads to the requirement that

$$u_\tau l/\nu > 6{\cdot}6/\sigma^{\frac{1}{2}}, \tag{6.1}$$

which must be satisfied for the law $Nu \propto \tau_w^{\frac{1}{3}}$ to apply (to within 5 % of τ_w). This result is equivalent to that of Liepmann & Skinner (1954) that the Nusselt number should be $\gg 1$, but is more explicit.

Condition (2) has been discussed for the turbulent boundary layer case in §5. It was found that the relationship

$$u_\tau l/\nu < 64\sigma \tag{6.2}$$

must be satisfied for a unique calibration in laminar and turbulent flows to be expected. (It is clear from the combination of these two conditions that, for turbulent boundary layers, the heated film technique is particularly suitable for fluids of high Prandtl number.)

For laminar boundary layers this condition is only significant (errors in skin friction less than 5 %) within 2 or 3 gauge lengths from a stagnation point and 6 gauge lengths from the leading edge for a flat plate flow. Errors are small near separation if the distance to the separation point exceeds approximately 50 gauge lengths. A more detailed discussion is given in Brown (1967*b*).

Condition (3) appears to be relatively unimportant since, in practice, l is an effective length found from calibration such that the actual heat supplied is equivalent to that from a 'top-hat' distribution. The analysis has also been carried out for an error function wall temperature distribution (using the Fourier transform) and the various constants found to differ very little from those presented here.

6.2. *Experimental application of the results*

It is fortuitous that the two series (4.10) and (4.14) relating the heat convected into the stream to skin friction and pressure gradient can both be represented rather accurately by the simple relationship

$$\tau_w + \frac{5}{18}\frac{dp}{dx}\frac{l}{Nu} = \frac{19}{10}\frac{\mu^2}{\rho\sigma l^2}Nu^3, \tag{6.3}$$

the form of which was found from the integral approach referred to in §4.3.

For the measurement of skin friction in flows with a pressure gradient the second term represents a correction to the relationship $Nu \propto \tau_w^{\frac{1}{3}}$, the latter being found by calibration in zero pressure gradient. An investigation of this correction term (Brown 1967*b*) shows that it is only important (correction less than 10 %) within 40 gauge lengths from the stagnation point in favourable pressure gradients, but in typical adverse pressure gradients it is of the order of 10 % of τ_w when the skin friction is 40 % of its Blasius value. A channel flow has also been considered, since this may be a useful way of calibrating the device, and it was found that the correction term was less than 2 % of τ_w provided $\alpha^{\frac{1}{3}}h/l > 28$, where h is the width of the channel.

Detailed experiments to measure skin friction in an adverse pressure gradient, using the heated film and Stanton tube for comparison, have been reported in Brown (1967*a*). It was found that a Stanton tube and a heated film gave similar values for skin friction in slight adverse pressure gradients (agreement within 10 %) but divergent values for an increasingly unfavourable gradient. Theoretical predictions of skin friction using Curle's (1962) method and measurements with the heated film (corrected for pressure gradient using (6·3) agreed closely (7 %) up to the point where the skin friction was 30 % of its Blasius value. At separation the experimental value of Stratford's (1954) parameter $x^2C_p(dC_p/dx)^2$ was less than that predicted by Curle's method, and if the method was modified accordingly, measurements and predictions of skin friction agreed within 10 % up to the point where skin friction was 10 % of its Blasius value. At this point the Stanton tube results appeared to be at least 200 % in error. It would seem that this increasing discrepancy between the two measuring techniques was due to the increasing curvature of the velocity profile near the wall which could be taken into account in the case of the heated film but not in the case of the Stanton tube.

An experiment was also reported (Brown 1967*a*) in which skin friction was measured in both turbulent and laminar channel flows with a heated film and a Stanton tube. Two, quite distinct, typical calibrations were found for the Stanton tube, and a single calibration found for the very short (0·006 in.) heated film used. The range of $u_\tau l/\nu$ in the experiment was between 12 and 40 which for air is within the limits described in §6.1.

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